Arc Length Calculator

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Abstract:

- The Arc Length formula is a definite integral, which is intended to approximate the length of a function f(x), over interval [a, b], if f(x) is differentiable across the interval. The graph of the function is divided up into an infinitesimal series of points, which are plotted along the curve of the function, from point a to point b. Each pair of points across that breadth is then connected by a straight line, so as to form a newfound series of rough right triangles against the graph, between each pair of points, with the drawn line acting as the hypotenuse. After finding $\Delta y / \Delta x$ between given points xn-1 and xn, it is possible to invoke the Pythagorean Theorem [a2 + b2 = c2] to solve for the length of the hypotenuse [c], with legs a & b representing the vertical and horizontal changes in the function between points xn-1 and xn. The sum of all the hypotenuses will be the approximate arc length of the function over [a, b]

Conventions:

- Finding the arc length of a function has multiple practical applications. For example, say you wanted to calculate the distance that you had to travel along a winding road from point a to point b. GPS pathfinding algorithms use satellite technology, and must analyze road networks, as well as the length of all those roads. Computing the distance travelled along a road is a wonderful example of a conventional use for the arc length equation. In order to accomplish this, you could imagine the road as a 2D function from interval [a, b], and then get a reasonably accurate estimate of the road's true length. Another pragmatic use would be estimating the distance travelled by a space shuttle across its parabolic trajectory. Essentially any scenario in which knowing the length of something that isn't wholly straight represents a good application of the arc length formula.

Problem Formation:

- In order to program an arc length calculator, I have elected to use Microsoft Excel. The first step is to take the Pythagorean Theorem (a^2 + b^2 = c^2), and through algebraic manipulation, morph it into the equation for the arc length of a function:

$$\frac{C^2}{A^2} = 1 + \frac{B^2}{A^2} \quad = \quad \frac{C}{A} = \sqrt{1 + \left(\frac{B}{A}\right)^2} \quad = \quad C = \sqrt{1 + \left(\frac{B}{A}\right)^2}\,A$$

If we replace the variables in the equation and integrate, we then get:

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \quad => \quad L = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\,dx$$

Next, we must select a value of n, which will represent how many subintervals the interval [a, b] will be divided into. Then, after specifying the lower-bound a and the upper-bound b, we will have to input a value for the dx, which represents the width of each subinterval. Finally, all we have left to do is plug in the value of f(x), and let the Excel calculator do the rest of the work for us. It will print out a numerical approximation of the arc length of the function, with a margin of error inversely proportional to the value of n (I.e., the more subintervals there are, the more accurate the approximation will be).

Results:

- Say we want to use my arc length calculator to find the arc length of the function: $y = \sin(x)$, from $x = 0$ to $x = \pi$
- First, we must recognize that the lower-bound a in this case is 0, and that the upper-bound b is π.
- Second, we will need to decide on a value of n. For the sake of this example, we will make it: 12
- Third, we will have to assign dx an appropriate value. Since we chose 12 as our value of n, we will make dx: π / 12

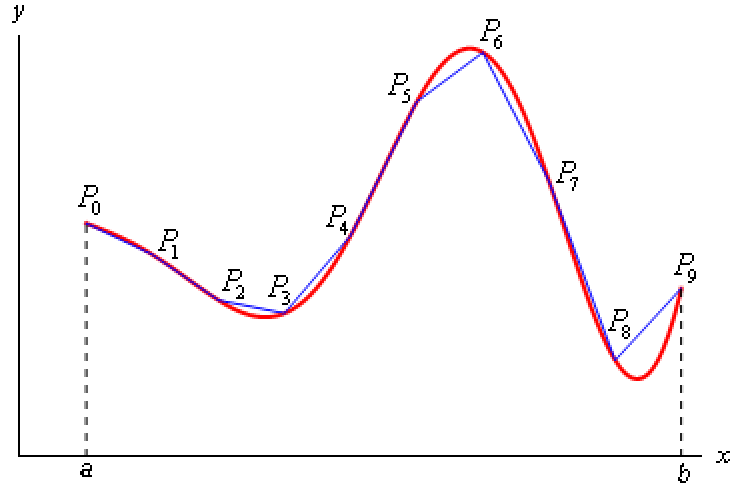
	A	B
1	a	0
2	b	3.14159
3	n	12
4	dx	0.261799
5		

- Lastly, we just have to punch in the value of f(x) and see the result. Below is the result:

	A	B	C	D	E	F	G
1	a	0			test pts	f(x)	distances
2	b	3.14159			0	0	
3	n	12			0.261799	0.258819	0.368138
4	dx	0.261799			0.523598	0.499999	0.355959
5					0.785397	0.707106	0.333814
6	Arc				1.047196	0.866025	0.306258
7	Length	3.816775			1.308995	0.965925	0.280212
8	Approx.				1.570794	1	0.264007
9					1.832593	0.965927	0.264007
10					2.094392	0.866027	0.280212
11					2.356191	0.707109	0.306257
12					2.61799	0.500003	0.333814
13					2.879789	0.258823	0.355959
14					3.141588	4.65E-06	0.368138
15							3.816775

Objectives:

The goal with calculating the arc length of a function is to get as close as possible to the actual length of the graph from starting point a, to ending point b. The graph below demonstrates how the formula for arc length actually works. By drawing a polygonal path along the curve, with n nodes in the path, we can simply measure the length of each line by using the change in x and y between a pair of points, and the Pythagorean Theorem. The more nodes or points that there are along the graph of a function, the better the estimate for the length of the function will be.



Example:

- Let us consider the following example from MyMathLab:

Find the length of the following curve.

$y = 3x^{\frac{3}{2}}$ from $x = 0$ to $x = 8$

To input such a problem to my arc length calculator, we must break up our input into four parts:

1.) Insert f(x), which in this case is: 3x^(3/2)

2.) Specify a, the lower-bound of our definite integral, which in this case is: 0

3.) Specify b, the upper-bound of our definite integral, which in this case is: 8

4.) Specify a particular even value of n to subdivide our interval [a, b] into n equal-sized pieces, which for the sake of this example will be: 8

5.) Specify the value for dx, which for the sake of this example will be: 1

Now that we have our input, it's time to solve:

a.) The numerical approximation for the arc length of the function is: 68.44779

	A	B	C	D	E	F	G
1	a	0			test pts	f(x)	distances
2	b	8			0	0	
3	n	8			1	3	3.162278
4	dx	1			2	8.485281	5.575689
5					3	15.58846	7.173222
6	Arc				4	24	8.470776
7	Length	68.44779			5	33.54102	9.593282
8	Approx.				6	44.09082	10.59708
9					7	55.56078	11.51347
10					8	67.88225	12.36199
11							68.44779
12							
13							

Conclusion:

- After going through the hoops that this project required me to go through, I can safely say that I have a renewed appreciation for the effort that programmers put into meticulously getting a calculator to accurately parse, graph, differentiate, and integrate any particular function that you – the user – inputs. I've greatly increased my aptitude for manually taking the derivative of a function, integrating a function, and specifically finding the arc length of a graph. This is all valuable information that will come in handy as I brave my final exam.

Bibliography/Sources:

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- https://www.purplemath.com/modules/distform.htm